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A MODIFIED HEINZ'S INEQUALITY

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Concerning the Heinz's inequality, we got the following result.

Theorem. The region of γ such that the operator inequality

$$(A^{\gamma}A^{\alpha}A^{\gamma})^{\beta} \geq (A^{\gamma}B^{\alpha}A^{\gamma})^{\beta}$$

holds for any operators A and B such as $A \geq B \geq bI$ (some $b > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$ is as follows ;

(1) $0 < \alpha \leq 1,\ 0 < \beta \leq 1, \qquad -\infty < \gamma < +\infty$

(2) $0 < \alpha \leq 1,\ 1 < \beta \leq 2,$

$$\max\left\{-\frac{1}{2},\ \frac{-\alpha\beta}{2(\beta-1)}\right\} \leq \gamma \leq \min\left\{0,\ \frac{1-\alpha\beta}{2(\beta-1)}\right\}$$

(3) $0 < \alpha \leq 1,\ 2 < \beta \leq \dfrac{1}{\alpha}, \qquad \gamma = 0$

(4) $0 < \alpha \leq 1,\ 2 < \beta,$

$$\max\left\{\frac{2\alpha-1-\alpha\beta}{2(\beta-1)},\ \frac{-\alpha\beta}{2(\beta-1)}\right\} \leq \gamma \leq \min\left\{\frac{2\alpha-\alpha\beta}{2(\beta-1)},\ \frac{1-\alpha\beta}{2(\beta-1)}\right\}$$

and (5) $1 < \alpha,\ 0 < \beta < 1,$

$$\max\left\{0,\ \frac{\alpha\beta-1}{2(1-\beta)}\right\} \leq \gamma.$$

In this lecture, we constituted counter examples of A and B for the cases where

$$0 < \alpha \leq 1, \quad 1 < \beta, \quad \min\left\{\min\left(0,\ \frac{2\alpha-\alpha\beta}{2(\beta-1)}\right),\quad \frac{1-\alpha\beta}{2(\beta-1)}\right\} < \gamma (\neq 0)$$

and

$$0 < \alpha \leq 1, \quad 1 < \beta, \quad \gamma < \max\left\{\max\left(-\frac{1}{2},\quad \frac{2\alpha-1-\alpha\beta}{2(\beta-1)}\right),\quad \frac{-\alpha\beta}{2(\beta-1)}\right\}$$

as follows.

For any a, b and μ such as $1 < a \leq \frac{17}{16}$, $0 < b \leq \frac{1}{2}$ and $0 < \mu \leq \min\{\alpha,\ 1-\alpha\}$, let

$$x = a + \frac{(a-1)b^{1-\mu}}{a-1+b(1-b)} - \frac{(a-1)\left(b^{2-\mu}+b^{1-\mu}\right)}{b(1-b)} - (1-b)b^{2-\alpha}. \qquad (\sharp_1)$$

Then

$$a - x = (1-b)b^{2-\alpha} + \frac{(a-1)\left[b^{2-\mu}\{a-1+b(1-b)\}+(a-1)b^{1-\mu}\right]}{b(1-b)\{a-1+b(1-b)\}} \qquad (\sharp_2)$$

and

$$1 - \frac{dx}{da} = \frac{(a-1)b^{1-\mu}\{a-1+2b(1-b)\}}{b(1-b)\{a-1+b(1-b)\}^2} + \frac{b^{2-\mu}}{b(1-b)}. \qquad (\sharp_3)$$

Since $\mu+2 > 2-\alpha$, $b^{\mu+2} < b^{2-\alpha}$ and, by taking $(1-b)b^{\mu+2}\ (< a-1\ )$ sufficiently near by $a-1$, we have $a-1 < (1-b)b^{2-\alpha}$ and hence we can choose b such as

$$(1-b)b^{\mu+2} < a-1 < (1-b)b^{2-\alpha}. \qquad (\sharp_4)$$

By $(\sharp_2)$ and $(\sharp_4)$, we have

$$\begin{aligned} & a-1 < (1-b)b^{2-\alpha} < a-x \\ & < (1-b)b^{2-\alpha} + \frac{(a-1)b^{1-\mu}}{1-b} + \frac{(a-1)^2b^{-1-\mu}}{(1-b)^2} \\ & < (1-b)b^{2-\alpha} + b^{3-(\mu+\alpha)} + b^{3-(\mu+2\alpha)} \to 0 \quad (\text{as } a \to 1) \end{aligned} \qquad (\sharp_5)$$

because $b \to 0$ (as $a \to 1$) by $(\sharp_4)$ and $o < (\mu+2\alpha) \leq \begin{cases} 3\alpha & (0 < \alpha \leq \frac{1}{2}) \\ 1+\alpha & (\frac{1}{2} \leq \alpha < 1) \end{cases} \leq 2.$

And, by $(\sharp_3)$ and by $(\sharp_4)$, we have

$$\begin{aligned} \frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} & < \frac{2(a-1)b^{1-\mu}}{b(1-b)\{a-1+b(1-b)\}} + \frac{b^{1-\mu}}{1-b} \\ & < \frac{2(a-1)b^{-1-\mu}}{(1-b)^2} + \frac{b^{1-\mu}}{1-b} \\ & < \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b} \end{aligned} \qquad (\sharp_6)$$

For any ϵ such as $0 < \epsilon < \frac{(a-b)(a-1)}{b(1-b)}$, let $\delta = \frac{b(1-b)\epsilon}{a-1}$. Then $0 < \delta < a-b$ and $a-b-\delta > 0$. And let

$$A = \begin{pmatrix} a & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & b+\epsilon+\delta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

where $y = b - \frac{\epsilon(a-b-\delta)}{a-1}$. Then A and B are self-adjoint and clearly $B \geq \min(x,\ y)\ I$. Since $x \to 1$ (as $a \to 1$) by $(\sharp_5)$, we may assume that $\min(x,\ y) > \frac{b}{2}$ because we shall consider the case where $\epsilon \to 0$ and nextly $a \to 1$. Since the proper polynomial of $A - B = \begin{pmatrix} a-x & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & \epsilon+\delta+\frac{\epsilon(a-b-\delta)}{a-1} \end{pmatrix}$ is

$$\lambda^2 - \left(a - x + \epsilon + \delta + \frac{\epsilon(a-b-\delta)}{a-1}\right)\lambda + (a-x)(\epsilon+\delta) + \left(\frac{a-x}{a-1} - 1\right)\epsilon(a-b-\delta)$$

and since $a - x > a - 1$ by ($\sharp_5$), $(a-x)(\epsilon+\delta) + \left(\frac{a-x}{a-1} - 1\right)\epsilon(a-b-\delta) > 0$ and $A \geq B$.

If $(A^\gamma A^\alpha A^\gamma)^\beta \geq (A^\gamma B^\alpha A^\gamma)^\beta$, then we have

$$\begin{aligned}
&a^{2\gamma}b^{2\gamma}(x^\alpha - y^\alpha)^2(a-b)\epsilon\left\{1 + \frac{o(\epsilon)}{\epsilon}\right\} \\
&\times\Bigg[\left\{a^{2\gamma\beta}x^{\alpha\beta} - a^{(\alpha+2\gamma)\beta}\right\} + a^{2\gamma\beta}\Bigg\{\frac{2\gamma(x^{\alpha\beta} - a^{\alpha\beta})}{a} + \frac{x^{\alpha\beta}(y^\alpha - x^\alpha)}{x^\alpha(a-b)} \\
&\qquad + \frac{x^{\alpha\beta}b^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)x^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})} - \alpha a^{\alpha\beta-1}\Bigg\}\beta\epsilon + o(\epsilon)\Bigg] \\
&\times b^{(\alpha+2\gamma)\beta}\Bigg[\Bigg\{\frac{\alpha(1-b)}{a-1} - \frac{x^\alpha - y^\alpha}{(a-b)y^\alpha} + \frac{\alpha(a-b)}{b(a-1)} \\
&\qquad + \frac{a^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)y^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})}\Bigg\}\beta\epsilon + o(\epsilon)\Bigg] \\
\leq\ &\frac{a^{4\gamma}b^{4\gamma}(x^\alpha - y^\alpha)^4\epsilon^2}{(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})^2}\left\{1 + \frac{o(\epsilon)}{\epsilon}\right\} \\
&\times\Bigg[\left\{a^{(\alpha+2\gamma)\beta} - y^{\alpha\beta}b^{2\gamma\beta}\right\} + \Bigg\{(\alpha+2\gamma)a^{(\alpha+2\gamma)\beta-1} - \Bigg(\frac{2\gamma(1-b)}{a-1} \\
&\qquad + \frac{x^\alpha - y^\alpha}{(a-b)y^\alpha} - \frac{a^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)y^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})}\Bigg)y^{\alpha\beta}b^{2\gamma\beta}\Bigg\}\beta\epsilon + o(\epsilon)\Bigg] \\
&\times\Bigg[\left\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}x^{\alpha\beta}\right\} + \Bigg\{\frac{(\alpha+2\gamma)(1-b)b^{(\alpha+2\gamma)\beta}}{a-1} - \Bigg(\frac{2\gamma}{a} \\
&\qquad + \frac{y^\alpha - x^\alpha}{x^\alpha(a-b)} + \frac{b^{2\gamma}(x^\alpha - y^\alpha)^2}{(a-b)x^\alpha(a^{2\gamma}x^\alpha - y^\alpha b^{2\gamma})}\Bigg)a^{2\gamma\beta}x^{\alpha\beta}\Bigg\}\beta\epsilon + o(\epsilon)\Bigg] \qquad \text{(i)}
\end{aligned}$$

where $o(\epsilon)$ is a function of ϵ such that $\lim_{\epsilon\to 0} \frac{o(\epsilon)}{\epsilon} = 0$.

By multiplying $\{a^{2\gamma(1+\beta)}b^{4\gamma}(x^\alpha - y^\alpha)^4\epsilon^2\}^{-1}$ to the both side of the inequality (i) and by putting $\epsilon \to 0$, we have

$$\begin{aligned}
&\frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha - b^\alpha)^2}\left\{\frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})}\right\} \\
\leq\ &\frac{a^{2\gamma(1-\beta)}}{(b^{\alpha+2\gamma} - a^{2\gamma}x^\alpha)^2}\left\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}x^{\alpha\beta}\right\} \qquad \text{(ii)}
\end{aligned}$$

and, by multiplying $b^{-2(\alpha+2\gamma)(\beta-1)}$ to the both side of (ii), we have

$$\begin{aligned}
&\frac{\beta(a^{\alpha\beta}-x^{\alpha\beta})b^{2\gamma(1-\beta)-\alpha\beta+2\alpha}}{(x^{\alpha}-b^{\alpha})^2}\left\{\frac{x^{\alpha}-b^{\alpha}}{b^{\alpha}}-\frac{\alpha(a-b^2)(a-b)}{b(a-1)}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{a^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{b^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\\
\leq\ &\frac{a^{2\gamma(1-\beta)}}{\{1-a^{2\gamma}x^{\alpha}b^{-(\alpha+2\gamma)}\}^2}\left\{a^{(\alpha+2\gamma)\beta}b^{-(\alpha+2\gamma)\beta}-1\right\}\left\{1-a^{2\gamma\beta}x^{\alpha\beta}b^{-(\alpha+2\gamma)\beta}\right\}. \qquad \text{(iii)}
\end{aligned}$$

Since $0 < a-x < 3b$ by $(\sharp_5)$, we have

$$\left|a^{\alpha\beta-1}-x^{\alpha\beta-1}\right| \leq Kb \text{ for some constant } K > 0 \qquad (\natural_1)$$

(case 1) Let $0 < \alpha \leq 1, \quad 1 < \beta, \quad \min\left\{\min\left(0,\ \frac{2\alpha-\alpha\beta}{2(\beta-1)}\right),\quad \frac{1-\alpha\beta}{2(\beta-1)}\right\} < \gamma(\neq 0)$.

In this case

$$\left\{\begin{aligned}
&(\alpha+2\gamma)\beta-2\gamma-\alpha-\min\{\min(\alpha(\beta-1),\ \alpha),\ 1-\alpha\}\\
&\qquad=2\gamma(\beta-1)+\alpha\beta-\alpha-\min\{\min(\alpha(\beta-1),\ \alpha),\ 1-\alpha\}>0\\
&\text{and}\quad(\alpha+2\gamma)(\beta-1)=(\alpha+2\gamma)\beta-2\gamma-\alpha\\
&\qquad\qquad\qquad\qquad>\min\{\min(\alpha(\beta-1),\ \alpha),\ 1-\alpha\}\geq 0\\
&\text{and hence}\quad\alpha+2\gamma>0\quad\text{because}\quad\beta>1
\end{aligned}\right. \qquad (\flat_1)$$

Let $a \to 1$. Then $b \to 0$ by $(\sharp_4)$ and $x \to 1$ by $(\sharp_5)$ and hence we have

$$\lim_{a\to1}\frac{a^{2\gamma(1-\beta)}}{(b^{\alpha+2\gamma}-a^{2\gamma}x^{\alpha})^2}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}x^{\alpha\beta}\right\}=-1$$

because $\alpha+2\gamma>0$ by $(\flat_1)$.

Since $0<\min\{\min(\alpha(\beta-1),\ \alpha),\ 1-\alpha\}\leq\min\{\alpha,\ 1-\alpha\}$, let

$$\mu=\min\{\min(\alpha(\beta-1),\ \alpha),\ 1-\alpha\}.$$

Then $(\alpha+2\gamma)\beta-2\gamma-\alpha-\mu>0$ by $(\flat_1)$ and since $b\to0$ (as $a\to1$) by $(\sharp_4)$ and $x\to1$ (as $a\to1$) by $(\sharp_5)$, we have

$$\begin{aligned}
\lim_{a\to1}\frac{a^{\alpha\beta}-x^{\alpha\beta}}{a-1}b^{(\alpha+2\gamma)\beta-2\gamma-1}&=\lim_{a\to1}\alpha\beta\left(a^{\alpha\beta-1}-x^{\alpha\beta-1}\frac{dx}{da}\right)b^{(\alpha+2\gamma)\beta-2\gamma-1}\\
&=\alpha\beta\lim_{b\to0}\left(1-\frac{dx}{da}\right)b^{(\alpha+2\gamma)\beta-2\gamma-1}=0 \text{ by } (\natural_1)
\end{aligned}$$

because $\frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} < \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b}$ by ($\sharp_6$) and hence we have

$$\lim_{a\to 1} \frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} = 0.$$

This contradicts (ii).

(case 2) Let $0 < \alpha \leq 1$, $1 < \beta$, $\gamma < \max\left\{\max\left(-\frac{1}{2}, \frac{2\alpha-1-\alpha\beta}{2(\beta-1)}\right), \frac{-\alpha\beta}{2(\beta-1)}\right\}$.

In this case

$$\begin{cases} 2\gamma(\beta-1) + \alpha\beta + 1 - \alpha - \max\{\max(\alpha\beta+2-\beta-\alpha,\ \alpha),\ 1-\alpha\} < 0 \\ \text{and} \quad (\alpha+2\gamma)(\beta-1) < -1 + \max\{\max(\alpha\beta+2-\beta-\alpha,\ \alpha),\ 1-\alpha\} \\ \qquad\qquad = -\min\{(1-\alpha)\min(\beta-1,\ 1),\ \alpha\} \leq 0 \\ \text{and hence} \quad \alpha + 2\gamma < 0 \quad \text{because} \quad \beta > 1 \end{cases} \tag{$\flat_2$}$$

Let $a \to 1$. Then $b \to 0$ by ($\sharp_4$) and $x \to 1$ by ($\sharp_5$) and hence we have

$$\lim_{a\to 1} \frac{a^{2\gamma(1-\beta)}}{\{1 - a^{2\gamma}x^\alpha b^{-(\alpha+2\gamma)}\}^2} \left\{a^{(\alpha+2\gamma)\beta} b^{-(\alpha+2\gamma)\beta} - 1\right\} \left\{1 - a^{2\gamma\beta}x^{\alpha\beta}b^{-(\alpha+2\gamma)\beta}\right\}$$
$$= -1$$

because $\alpha + 2\gamma < 0$ by ($\flat_2$).

Since $0 < \min\{(1-\alpha)\min(\beta-1,\ 1),\ \alpha\} \leq \min\{\alpha,\ 1-\alpha\}$, let

$$\mu = \min\{(1-\alpha)\min(\beta-1,\ 1),\ \alpha\}.$$

Then $2\gamma(1-\beta) - \alpha\beta + \alpha - \mu > 0$ by ($\flat_2$) and since $b \to 0$ (as $a \to 1$) by ($\sharp_4$) and $x \to 1$ (as $a \to 1$) by ($\sharp_5$), we have

$$\begin{aligned} &\lim_{a\to 1} \frac{a^{\alpha\beta} - x^{\alpha\beta}}{a-1} b^{2\gamma(1-\beta)-\alpha\beta+2\alpha-1} \\ &= \lim_{a\to 1} \alpha\beta \left(a^{\alpha\beta-1} - x^{\alpha\beta-1}\frac{dx}{da}\right) b^{2\gamma(1-\beta)-\alpha\beta+2\alpha-1} \\ &= \alpha\beta \lim_{b\to 0} \left(1 - \frac{dx}{da}\right) b^{2\gamma(1-\beta)-\alpha\beta+2\alpha-1} = 0 \text{ by } (\natural_1) \end{aligned}$$

because $\frac{b^{1-\mu}}{1-b} < 1 - \frac{dx}{da} < \frac{2b^{1-(\mu+\alpha)}}{1-b} + \frac{b^{1-\mu}}{1-b}$ by ($\sharp_6$) and hence we have

$$\lim_{a\to 1} \frac{\beta(a^{\alpha\beta} - x^{\alpha\beta})b^{2\gamma(1-\beta)-\alpha\beta+2\alpha}}{(x^\alpha - b^\alpha)^2} \left\{ \frac{x^\alpha - b^\alpha}{b^\alpha} - \frac{\alpha(a-b^2)(a-b)}{b(a-1)} - \frac{a^{2\gamma}(x^\alpha - b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha - b^{\alpha+2\gamma})} \right\} = 0.$$

This contradicts (iii).